Varying speed of light cosmology from a stringy short distance cutoff

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arXiv:astro-ph/0111479 v1 26 Nov 2001

It is shown that varying speed of light cosmology follows from a string-inspired minimal length uncertainty relation. Due to the reduction of the available phase space volume per quantum mode at short wavelengths, the equation of state of ultrarelativistic particles stiffens at very high densities. This causes a stronger than usual deceleration of the scale factor which competes with a higher than usual propagation speed of the particles. Various measures for the effective propagation speed are analyzed: the group and phase velocity in the high energy tail, the thermal average of the group and phase velocity, and the speed of sound. Of these three groups, only the first provides a possible solution to the cosmological horizon problem.

I. INTRODUCTION

Varying speed of light (VSL) cosmology was conceived as an alternative solution to the cosmological horizon problem [1, 2, 3, 4]. In contrast with inflation, where the comoving horizon scale shrinks as a consequence of the accelerating growth of the scale factor, VSL keeps the scale factor dynamics of the standard hot big bang unchanged but assumes that the speed of light, $c(t)$, is a decreasing function of of cosmic time t. The horizon problem can be solved provided that $c(t)$ drops sufficiently steeply for some time. In the original formulation, $c(t)$ was treated as a free parameter of the theory.

In Ref. [5], it was noted that VSL can effectively be obtained from the thermodynamical properties of theories with nonlinear dispersion relations which, in turn, were motivated in [5] by noncommutative geometry. While there are indications from string/M-theory that noncommutativity of spatial coordinate operators may be relevant near the string scale, we are lacking a simple, concrete realization of noncommutative geometry that can be used in the cosmological context. Hence one is forced to reduce its effects to a modified dispersion relation.

A very similar yet more specific approach is taken in this work. The starting point is an observation that was made in numerous studies of quantum gravity and string theory: under rather general assumptions, one can show that there exists a minimal length scale that can be probed by experiments whereas the resolution of shorter distances is prohibited by quantum gravitational effects (e.g., [6, 7, 8, 9]). Examined from the point of view of the low energy effective theory, it appears that quantum gravity modifies the Heisenberg uncertainty relation to

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta(\Delta p)^2 + \ldots\right) \qquad (1)$$

which gives rise to a minimal short distance uncertainty $\Delta x_{\rm min} = \hbar\sqrt{\beta}$ even for infinite Δp. For practical purposes, $\Delta x_{\rm min}$ can be identified with either the string scale or the Planck length, where the former could be a few orders of magnitudes larger than the latter. The correction may of course originate from complicated dynamics of the underlying fundamental theory. On the other hand, it can be modeled by equivalent corrections to the canonical commutation relation:

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \beta\mathbf{p}^2 + \ldots) \qquad (2)$$

as discussed in [10]. Indeed, the uncertainty relation above belongs to class of only very few types of short-distance structures of space-time that are admitted under very general assumptions [11]. Hilbert space representations of Eq. 2 were employed, for instance, for regularizing field theory [12] and, more recently, for analyzing the impact of short distance uncertainty on the predictions of inflation [13, 14].

The finite spatial localization of particles also changes their thermodynamic behavior at very high density. This has already been observed in [15, 16] where some cosmological consequences have been pointed out. However, a complete analysis of this theory in the context of VSL cosmology has not been done and shall be provided in this work. The most relevant features of the minimal length uncertainty theory are summarized in Sec. II, and are used to derive the expressions for the energy density and pressure of a radiation fluid in Sec. III. In Sec. IV, the cosmological implications are discussed and a further outlook is presented in Sec. V.

II. QUANTUM THEORY WITH MINIMAL SHORT DISTANCE UNCERTAINTY

An explicit example for the choice of commutation relations in three spatial dimensions that maintains translation and rotation invariance is given in [12]:

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\left(\frac{\beta\mathbf{p}^2}{(1+2\beta\mathbf{p}^2)^{1/2} - 1}\delta_{ij} + \beta\mathbf{p}_i\mathbf{p}_j\right) \qquad (3)$$

$$[\mathbf{x}_i, \mathbf{x}_j] = 0 \qquad (4)$$

$$[\mathbf{p}_i, \mathbf{p}_j] = 0 \qquad (5)$$

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Demanding translation and rotation invariance, this choice is unique to first order in β [1]. It does, however, violate Lorentz invariance and thereby specify a preferred frame which, in the present context, coincides with the cosmological rest frame.

Owing to the finite short distance uncertainty, the usual Hilbert space representation in terms of positions eigenstates is no longer available (in contrast with the momentum representation $|p\rangle$, which still is). An alternative representation is found by introducing the translators $\mathbf{T}_i$ with eigenstates $|\kappa\rangle$ obeying $\mathbf{T}_i|\kappa\rangle = \kappa_i/i\hbar|\kappa\rangle$, $i = 1\ldots 3$, and $\kappa^2 < 2/\beta$ [12]. The projection onto momentum space is given by

$$\langle\kappa|p\rangle = \delta\left(p_i - \frac{\kappa_i}{1-\beta\kappa^2/2}\right) \ . \tag{6}$$

In terms of $\psi(\kappa) = \langle\kappa|\psi\rangle$, the operator representations and the scalar product become:

$$\mathbf{x}_i\psi(\kappa) = i\hbar\partial_{\kappa_i}\psi(\kappa) \tag{7}$$

$$\mathbf{p}_i\psi(\kappa) = \frac{\kappa_i}{1-\beta\kappa^2/2}\psi(\kappa) \tag{8}$$

$$\langle\psi_1|\psi_2\rangle = \int\limits_{\kappa^2<2/\beta} \mathrm{d}^3\kappa\ \psi_1^*(\kappa)\psi_2(\kappa) \ . \tag{9}$$

The same representation was employed in [13, 14].

As demonstrated in [12], it is now possible to construct the closest analogue to the position representation by replacing the usual position eigenstates with states of maximal localization around a given position, $|x^{\mathrm{ml}}\rangle$. Their κ-space representation can be constructed by applying the translation operator $e^{x\mathbf{T}}$ to the maximally localized field around the origin, and therefore it varies as $\sim e^{-ix\kappa/\hbar}$. Using (6), one finds the quasi-position representation of the plane wave with momentum p:

$$\langle p|x^{\mathrm{ml}}\rangle = N(p^2)\ \exp\left(\frac{-ix}{\hbar}\frac{p\sqrt{1+2\beta p^2}-1}{\beta p^2}\right) \ , \tag{10}$$

where $N(p^2)$ is a normalization coefficient. It is now clear that $\kappa/\hbar$ acts as a substitute of the usual wavenumber k, with the crucial difference that it only allows a minimum wavelength $\lambda_{\min} = \pi\hbar(2\beta)^{1/2}$ for $p \to \infty$, as required at the onset. This will become important for the computation of the thermodynamic variables in the next section.

III. EQUATION OF STATE OF A RADIATION FLUID

In order to derive the energy density and pressure of an ideal gas of photons (with straightforward generalization to other ultrarelativistic particles), the grand canonical formalism can be used (e.g. [17]). The grand potential is defined as

$$\Omega = -k_{\mathrm{B}}T\ \ln\mathcal{Z} \ , \tag{11}$$

where $\mathcal{Z}$ is the grand partition function,

$$\mathcal{Z} = \mathrm{Tr}\, e^{-(\mathbf{H}-\mu\mathbf{N})/k_{\mathrm{B}}T} \ , \tag{12}$$

$\mathbf{H}$ is the Hamiltonian, $\mathbf{N}$ is the number operator, k_{B} is Boltzmann's constant, and T is the temperature. Evaluating the trace in the number representation for bosons, one finds as usual

$$\Omega = k_{\mathrm{B}}T\sum_l \ln\left(1 - e^{-(E^l-\mu)/k_{\mathrm{B}}T}\right) \ . \tag{13}$$

The summation is over all allowed momentum states p^l and $E^l = c|p^l|$ is the energy of a relativistic particle in this state.

At this point, one usually transforms the sum into an integral over the three-dimensional wavenumber k by partitioning space into boxes of volume $V = L^3$ and demanding L-periodicity of plane waves with momentum $p = \hbar k$ in the position representation. In the present framework, a position representation is unavailable. Instead, the continuum limit is now most conveniently taken in κ-space: requiring the periodicity of $e^{ix\kappa/\hbar}$ yields $\kappa = 2\pi\hbar l/L$, where $l \in M$ labels the discrete quantum state and $M = \{l \in \mathbb{Z}^n\,|\,l^2 < (L/\lambda_{\min})^2\}$. For sufficiently large V, the energy levels are nearly continuous and Eq. (13) can be written as

$$\Omega = \frac{g\,V\,k_{\mathrm{B}}T}{(2\pi\hbar)^3}\int\limits_{\kappa^2<2/\beta}\mathrm{d}^3\kappa\ \ln\left(1 - e^{-[E(\kappa)-\mu]/k_{\mathrm{B}}T}\right) \tag{14}$$

with $E(\kappa) = c|p(\kappa)|$ and $g = 2$ for photons. After transforming the integral to energy space and writing it in terms of the dimensionless variables $\epsilon = E/E_\beta$, $\tilde{\mu} = \mu/E_\beta$, and $\tau = k_{\mathrm{B}}T/E_\beta$, where $E_\beta = c\beta^{-1/2}$ is the characteristic energy scale where quantum gravitational effects become relevant, one obtains

$$\Omega = \frac{4\pi\,g\,V\,E_\beta^4\,\tau}{(2\pi\hbar\,c)^3}\int\limits_0^\infty \mathrm{d}\epsilon\ J(\epsilon)\,\epsilon^2\,\ln\left(1 - e^{-[\epsilon-\tilde{\mu}]/\tau}\right) \ . \tag{15}$$

Here, J is the Jacobian determinant of the transformation $\kappa_i \to p_i = \kappa_i(1-\beta\kappa^2/2)^{-1}$, $p_i \in \mathbb{R}$, given in terms of ϵ by

$$J(\epsilon) = \frac{2}{\epsilon^6}\left(2 + \epsilon^2 - \frac{2+3\epsilon^2}{\sqrt{1+2\epsilon^2}}\right) \ . \tag{16}$$

$J(\epsilon)$ essentially contains all of the information about the modification of the phase space volume that results from the short distance cutoff.

One can now derive the thermodynamic variables from the grand potential in the standard way. Using $\rho(\tau) =$

[1] Note that in [15], a different choice for the commutation relations was made which is noncommutative in the spatial coordinates and breaks translation invariance.

$V^{-1}\,\partial_{\tau^{-1}}\,\tau^{-1}\Omega$ for the energy density, $P(\tau) = -\partial_V\,\Omega$ for the pressure, $N(\tau) = -V^{-1}\,\partial_\mu\,\Omega$ for the number density and setting $\tilde{\mu} = 0$ for ultrarelativistic particles, one finds:

$$\rho(\tau) \;=\; \frac{4\pi\,g\,E_\beta^4}{(2\pi\hbar\,c)^3}\int_0^\infty \mathrm{d}\epsilon\; J(\epsilon)\,\frac{\epsilon^3}{e^{\epsilon/\tau}-1} \qquad (17)$$

$$P(\tau) \;=\; -\frac{4\pi\,g\,E_\beta^4\,\tau}{(2\pi\hbar\,c)^3}\int_0^\infty \mathrm{d}\epsilon\; J(\epsilon)\,\epsilon^2\,\ln\left(1-e^{-\epsilon/\tau}\right) \qquad (18)$$

$$N(\tau) \;=\; \frac{4\pi\,g\,E_\beta^3}{(2\pi\hbar\,c)^3}\int_0^\infty \mathrm{d}\epsilon\; J(\epsilon)\,\frac{\epsilon^2}{e^{\epsilon/\tau}-1} \qquad (19)$$

It is useful to study the asymptotic behavior of the integrals for very small and very large τ (e.g., [18]). In the case of $\rho(\tau)$, rescaling the integrand by substituting $u = \epsilon/\tau$ shows that the dominant contribution comes from $u = O(1)$ for $\tau \to 0$. One can therefore write $J = 1+O(u^2\tau^2)$ and find the usual result $\rho(\tau) \sim \tau^4$ for $\tau \to 0$, as expected. For large τ, we first note that $J(\epsilon)$ drops as ϵ^{-4} for large ϵ. Hence, for $\epsilon = O(1)$ the integrand is $O(\tau)$ and so is the integral, whereas for $\epsilon = O(\tau)$, the integrand is $O(\tau^{-1})$ and the integral is $O(1)$. The dominant contribution therefore comes from $\epsilon = O(1)$ so that the exponential can be approximated by $1 + \epsilon/\tau + O(\epsilon^2\tau^{-2})$, yielding

$$\rho_{\tau\to\infty}(\tau) \simeq \frac{4\pi\,g\,E_\beta^4}{(2\pi\hbar\,c)^3}\,\tau\int_0^\infty \mathrm{d}\epsilon\; J(\epsilon)\,\epsilon^2\;, \qquad (20)$$

where the integral evaluates to $2^{3/2}/3$.

In the case of $P(\tau)$, partial integration yields

$$P(\tau) = \frac{4\pi\,g\,E_\beta^4}{3\,(2\pi\hbar\,c)^3}\int_0^\infty \mathrm{d}\epsilon\; G(\epsilon)\,\frac{\epsilon^3}{e^{\epsilon/\tau}-1}\;, \qquad (21)$$

where $G(\epsilon)$ is defined as

$$\begin{aligned} G(\epsilon) \;&=\; \frac{3}{\epsilon^3}\int \mathrm{d}\epsilon\; J(\epsilon)\,\epsilon^2 \\ &=\; \sqrt{1+2\epsilon^2}\;J(\epsilon)\;. \end{aligned} \qquad (22)$$

For small τ, $G(\epsilon) = 1 - O(\epsilon^2)$ behaves like $J(\epsilon)$, and one obtains the expected result $P = \rho/3$. On the other hand, $G \sim \epsilon^{-3}$ drops more slowly for $\tau \to \infty$ than $J(\epsilon)$. It is only for this reason that the ratio of pressure and energy density, and hence the dynamics of the scale factor, differs from its usual behavior at high temperatures, as will be demonstrated in Sec. IV. Indeed, for $\epsilon = O(\tau)$, the integrand is now $O(1)$ and the integral is $O(\tau)$, contributing equally strongly to the integral as $\epsilon = O(1)$. In this case, a good approximation is given by

$$\begin{aligned} P_{\tau\to\infty}(\tau) \;&\sim\; \int_{O(1)}^{O(\tau)} \mathrm{d}\epsilon\;\frac{\tau}{\epsilon} \\ &=\; \tau\,(A + B\log\tau)\;. \end{aligned} \qquad (23)$$

The cosmological evolution of ρ is determined, via the energy conservation equation, by the equation of state parameter $w = P/\rho$. For small τ, one recovers the usual equation of state, $w = 1/3$, whereas $\tau \to \infty$ gives $w_{\tau\to\infty}(\rho) = C + D\log\rho_{\tau\to\infty}$. The same form of w was discovered and analyzed in [5], hence one can expect a similar result for the cosmological evolution. As shown in Sec. IV, this is indeed true. The result for $w(\rho)$ was confirmed numerically by inverting $\rho(\tau)$ in order to evaluate $P(\tau)$. A good high-density fit is provided by $C \simeq -1.2$ and $D \simeq 2.3$.

Finally, using the same arguments as for $\rho(\tau)$, the expression for $N(\tau)$ yields the standard result for $\tau \to 0$ while $\tau \to \infty$ gives

$$N_{\tau\to\infty}(\tau) \simeq \frac{4\pi\,g\,E_\beta^3}{(2\pi\hbar\,c)^3}\,\tau\int_0^\infty \mathrm{d}\epsilon\; J(\epsilon)\,\epsilon = \frac{2\pi\,g\,E_\beta^3}{(2\pi\hbar\,c)^3}\,\tau\;. \qquad (24)$$

All of the preliminaries are now in place to examine the cosmological impact of the modified equation of state, which will proceed in close analogy with Ref. [5].

IV. COSMOLOGICAL IMPLICATIONS

The evolution of $\rho(t)$ and the scale factor $a(t)$ in the very early universe are determined by the Friedmann and energy conservation equations with negligible curvature and cosmological constant terms, given by

$$\left(\frac{\dot{a}}{a}\right)^2 \;=\; \frac{\rho}{3M_{\rm Pl}^2} \qquad (25)$$

$$\frac{\dot{\rho}}{\rho} \;=\; -\frac{3\dot{a}}{a}\,(1+w(\rho))\;, \qquad (26)$$

where a dot corresponds to a derivative with respect to cosmic time and $M_{\rm Pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass. These equations can only be solved numerically (starting with the known low-temperature solution and integrating backward), using the numerical evaluation of $w(\rho)$. However, if $w(\rho)$ varies sufficiently slowly with ρ, one can approximate the solution for the scale factor by the usual one for constant w, with w replaced by $w(\rho)$ and an offset for the origin of the time coordinate t_0:

$$a(t) \simeq (t-t_0)^{2/3(1+w(\rho))}\;. \qquad (27)$$

Fig. 1 demonstrates that this is a good approximation at all times except for the brief period where w transitions from $1/3$ to the logarithmic high-temperature behavior.

Eq. (27) captures the relevant scale factor dynamics in our model: at very high density, the deceleration $\sim \ddot{a}/aH^2$ is very large, giving rise to a smaller than usual Hubble rate $H = \dot{a}/a$, whereas the standard scale factor evolution is recovered at densities far below the string scale.

The high-density evolution of ρ as a function of the scale factor can be obtained directly by solving Eq. (26)

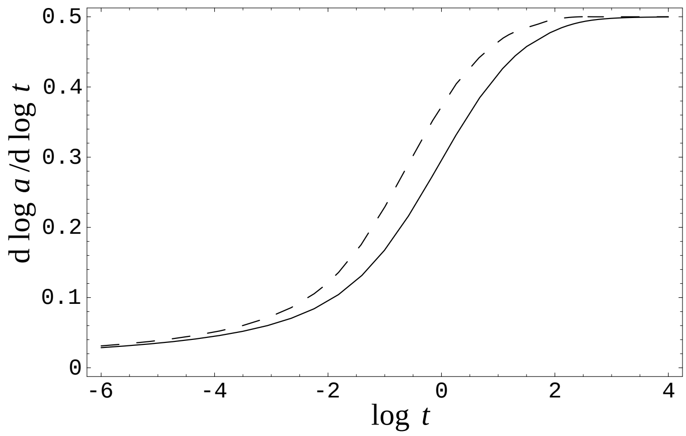

FIG. 1: Comparison of the effective time exponent of the numerical solution (solid line) and the approximation $2/3(1+w(\rho))$ (dashed line) as a function of time, adjusted for a shift of the time of the cosmic singularity.

with $w(\rho) \simeq w_{\tau\to\infty}$, yielding

$$\log \rho_{\tau\to\infty}(a) = E\, a^{-3D} - \frac{C+1}{D} \; . \tag{28}$$

This was also shown in [5].

The question whether the cosmological horizon problem can be solved in this model hinges crucially on the choice for the effective propagation speed of information, $c_{\rm eff}$. The VSL mechanism works if the comoving horizon scale,

$$r_{\rm h} = \frac{c_{\rm eff}}{aH} = \frac{c_{\rm eff}}{\dot a} \; , \tag{29}$$

declines for some period of time before turning around to its current growing behavior.

A case can be made that the most suitable measure for $c_{\rm eff}$ is given by the speed of sound, $c_{\rm s}$, of the radiation fluid rather than by the group or phase velocity of individual photons. It is given by

$$c_{\rm s}^2 = \left(\frac{\partial P}{\partial \rho}\right)_{S=const} = w + \rho \frac{\partial w}{\partial \rho} \; . \tag{30}$$

In the high-density limit, $c_{\rm s}$ scales like $(\log \rho)^{1/2} \sim a^{-3/2}$, whereas $\dot a \sim a \rho^{1/2} \sim a \exp(1/2a^3)$ which clearly wins as $a \to 0$. Consequently, $r_{\rm h}$ is always a growing function of time in this case and the horizon problem remains.

The group and phase velocities are given by

$$c_{\rm g}(\epsilon) = \frac{\partial E}{\partial |\kappa|} \; = \; \frac{c}{2}\left(1 + 2\epsilon^2 + \sqrt{1+2\epsilon^2}\right) \tag{31}$$

$$c_{\rm ph}(\epsilon) = \frac{E}{|\kappa|} \; = \; \frac{c}{2}\left(1 + \sqrt{1+2\epsilon^2}\right) \; . \tag{32}$$

As pointed out in [5], the evaluation of ϵ in these expressions is somewhat ambiguous. The bulk of all photons will populate a region around the peak in the modified Planck distribution, which saturates at $\tau \sim 1$. After saturation, the group and phase velocities of these photons will cease growing as a function of increasing density. However, the authors of [5] argue that the "fast tail" of the Planck distribution may provide the required causal contact, and they propose to evaluate $c_{\rm g}$ and $c_{\rm ph}$ at $\epsilon \simeq \tau \simeq \rho_{\tau\to\infty}$. In this case, $c_{\rm g} \sim \rho^2$ and $c_{\rm ph} \sim \rho$ at high densities, so that $r_{\rm h}$ indeed declines as a function of time (consistent with [5]). Consequently, this choice of effective velocity solves the horizon problem, provided that the model assumptions made in Sec. II remain valid well into the regime $\tau \gg 1$.

As a final possibility, one can consider a thermal average of $c_{\rm g}$ (or $c_{\rm ph}$), defined for instance as the particle number weighed value:

$$\langle c_{\rm g,ph}\rangle(\tau) \; = \; N^{-1}(\tau)\, \frac{4\pi\, g\, E_\beta^3}{(2\pi\hbar\, c)^3} \int_0^\infty {\rm d}\epsilon\, \frac{K_{\rm g,ph}(\epsilon)\, \epsilon^2}{e^{\epsilon/\tau} - 1} \tag{33}$$

$$K_{\rm g,ph}(\epsilon) \; = \; c_{\rm g,ph}\, J(\epsilon) \; . \tag{34}$$

The asymptotic properties of $N(\tau)$ were discussed in Sec. III. Noting that $K_{\rm g,ph} = 1 - O(\epsilon^2)$ for small ϵ one finds $\langle c_{\rm g,ph}\rangle(\tau) \simeq c$ as required. On the other hand, $K_{\rm g} \sim \epsilon^{-2}$ for large ϵ, whereas $K_{\rm ph} \sim \epsilon^{-3}$. The situation for the integral over $K_{\rm g}$ at $\tau \to \infty$ is therefore analogous to that of $P_{\tau\to\infty}$, yielding the same logarithmic growth. The normalization by $N(\tau)$ gives rise to the same behavior as found earlier for $w_{\tau\to\infty}$, i.e. $\langle c_{\rm g}\rangle_{\tau\to\infty} \sim \log \rho$. In contrast, the integral over $K_{\rm ph}$ at $\tau \to \infty$ is analogous to that of $\rho_{\tau\to\infty}$, hence the normalized result for $\langle c_{\rm ph}\rangle_{\tau\to\infty}$ asymptotes to a constant independent of τ. Explicitly, one finds $\langle c_{\rm ph}\rangle_{\tau\to\infty} = 2c$.

To summarize, the thermally averaged group or phase velocities grow at best logarithmically. This is insufficient to solve the horizon problem by means of the VSL mechanism, as argued above for $c_{\rm s}$.

Fig. 2 shows the size of the comoving horizon as a function of time, computed numerically using the "fast tail" group velocity and the thermally averaged one.

V. SUMMARY AND OUTLOOK

The analysis presented in this work is based on a rather simple premise: The spatial separation that can be probed by high-energy particles has a finite minimum value. Without referring to any specific quantum gravitational effect in particular, it is a very general model for the physics of relativistic particles at densities that occurred in the very early universe. It is very interesting that such a simple modification may provide an alternative solution to the horizon problem without resorting to the inflationary paradigm. Of course, one must now think of possibilities to resolve other quandaries that inflation has proved to be so helpful with, such as the flatness and relics problems and, perhaps most importantly, the generation of scale invariant fluctuations. Some ideas in this

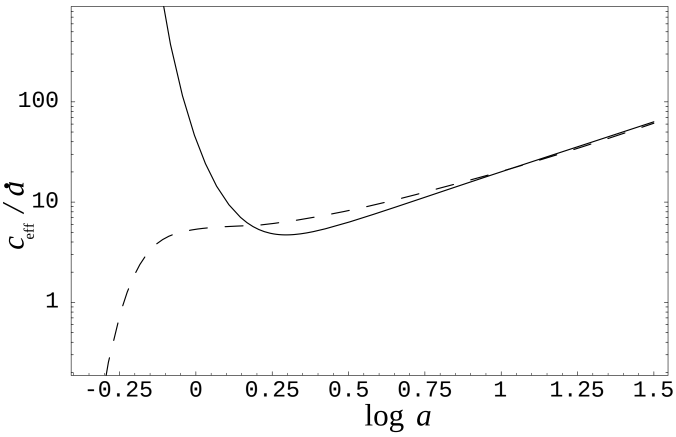

FIG. 2: Comoving horizon distance as a function of scale factor for the "fast tail" group velocity $c_{\rm g}$ (solid line) and the thermally averaged one, $\langle c_{\rm g}\rangle$ (dashed line). While the former can solve the horizon problem, the latter cannot. At late times, both converge onto the radiation dominated solution $\sim t^{1/2}$.

direction have already been put forward in the context of VSL [3, 5], and they can be readily generalized to the model in this work.

As always, many open questions remain. One of them, which may be answered by a more detailed analysis of transport processes in the early universe, is the correct choice of the effective communication speed. On a more fundamental level, one may question the validity of the minimal distance uncertainty principle as early as may be required to solve the horizon problem. In other words, it is possible (and even likely) that additional, quantum gravitational degrees of freedom will be excited in the relevant regime of ρ that will drastically change the equation of state. Unfortunately, the answer to this question might remain elusive until the fundamental theory is known in full detail.

Acknowledgments

I am happy to thank Achim Kempf and Joao Magueijo for valuable comments.

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